

SIMPLE BOL LOOPS

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Abstract. In this paper we investigate the Bol loops and connected with them groups. We prove an analog of the Doro's theorem for Moufang loops and find a criterion for simplicity of Bol loops. One of the main results obtained is the following: if the right multiplication group of a connected finite Bol loop S is a simple group, then S is a Moufang loop.

1. Introduction

Recall that a *loop* is a binary system S with unit element such that the equations $ax = b$ and $ya = b$ are uniquely solvable for all $a, b \in S$. The *Bol loops* are distinguished from the class of all loops by the identity

$$((xy)z)y = x((yz)y)$$

A Bol loop is called a *Moufang loop* if it satisfies the identity

$$y(z(yx)) = (y(zy))x.$$

These identities arose in connection with the so-called Bol closure conditions in web theory. The example $x \cdot y = x - y$ on \mathbb{R} shows that both identities in a quasigroup are independent and do not imply the existence of an identity element. As well known (see, e.g., [1]), every Bol loop is monoassociative and right invertible, and a left invertible Bol loop is Moufang.

A Bol loop is *simple* if it has no (nontrivial) proper homomorphic images, or equivalently, if it has no proper normal subloops. Let S be a finite simple Bol loop. Of course if S is associative, then S is a simple group. The classification of finite simple groups is known [2]. This is either a cyclic group of prime order or an alternating group of degree $n \geq 5$ or a group of Lie type or one of the twenty-six sporadic groups. All finite simple nonassociative Moufang loops are also known [3,4]. Every such loop is a homomorphic image of the loop of all invertible elements of split Cayley-Dickson algebra over $GF(q)$.

It is generally agreed that the most significant open problem in loop theory today is the existence of finite, simple Bol loops which are not Moufang. While we certainly have not solved these problems, we take a step in this direction. Let S be a Bol loop and $Gr_r(S)$ be its right multiplication group. We'll call the loop

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S *strongly simple* if the group $Gr_r(S)$ is simple. It is easy to prove (see [5]) that any strongly simple loop is simple. If S is infinite strongly simple Bol loop, then it may be non-Moufang. The corresponding example was constructed in [6]. If S is finite, then we have the following assertion:

Theorem. *Every connected finite strongly simple Bol loop is Moufang. In particular, if the right multiplication group $Gr_r(S)$ of a Bol loop S is a finite simple group of Lie type, then S is a finite simple Moufang loop.*

The present paper is organized as follows. In Sec. 2 we consider the typical situation of the abstract theory of symmetric spaces: a group G equipped with an involutory automorphism and the structure of the symmetric space on G defined by this pair. To this symmetric space one naturally assigns a symmetric space S , and sufficient conditions are found under which the natural binary composition transforms S into a Bol loop (or a Moufang loop). In Sec. 3, among the groups that are lead to a given Bol loop by using this construction, we choose an universal group from which the other groups are constructed as its homomorphic images. This makes it possible to give a criterion for simplicity of Bol loops S . In the last section, we prove the theorem.

2. A construction of Bol loops

Recall (see, e.g., [7]) that a binary system S is called a *symmetric space* if it satisfies the identities

$$\begin{aligned} x.x &= x, \\ x.(x.y) &= y, \\ x.(y.z) &= (x.y).(x.z) \end{aligned}$$

A symmetric space with the distinguished point $e \in S$ is called a *punctured symmetric space*, and the point e is referred to as the basic point of the space S .

The isomorphisms and homomorphisms of symmetric spaces are defined in the usual way; in the case of punctured spaces, it is assumed in addition that any homomorphism takes the basic point to the basic point. By induction one can define the powers of an element x (with respect to the basic point e): $x^0 = e$, $x^1 = x$, $x^{n+2} = x.(e.x^n)$, and $x^{-n} = e.x^n$. It is obvious that the map $x \rightarrow x^{-1}$ is an automorphism of a symmetric space.

Suppose G is a group admitting an involutory automorphism σ , I_σ is the set of fixed points of σ , and $G_\sigma = \{x^{-1}x^\sigma \mid x \in G\}$. It is easy to prove, that the set G_σ and the set of right cosets G/I_σ of the subgroup I_σ in the group G is a symmetric spaces with respect to the products

$$\begin{aligned} x.y &= xy^{-1}x, \\ I_\sigma x \cdot I_\sigma y &= I_\sigma y^\sigma (x^\sigma)^{-1}x. \end{aligned}$$

Moreover, the mapping $\psi : G/I_\sigma \rightarrow G_\sigma$ given by the relation $\psi(I_\sigma x) = x^{-1}x^\sigma$ defines an isomorphism $G/I_\sigma \simeq G_\sigma$ of the symmetric spaces.

Let N be the subgroup of G generated by the set G_σ . Suppose that there is a group homomorphism $\varphi : N \rightarrow G$ from N into G such that

- 1) $G = \langle x, x^\sigma \mid x \in \varphi(G_\sigma) \rangle$,
- 2) $\varphi(\varphi(x)^{-1}\varphi(x)^\sigma) = \varphi(x)$ for any $x \in G_\sigma$,
- 3) every coset of $I = I_\sigma \cap \varphi(N)$ in the group $\varphi(N)$ contains at least one element of $\varphi(G_\sigma)$.

(Here and everywhere below, the notation $G = \langle M \rangle$ means that the group G is generated by the set M). In this case, we say that G is a *group with the (σ, φ) -property* or a *(σ, φ) -group*. Moreover, everywhere below, we denote by the symbols N and I the corresponding subgroups of a (σ, φ) -group G and denote the images $\sigma(H)$ and $\varphi(H)$ of the subgroups H in G by the symbols H^σ and H^φ .

It is easily shown that for each coset Ig there exists exactly one element of $\varphi(G_\sigma)$. Indeed, if $Ig = Ih$, then $g^{-1}g^\sigma = h^{-1}h^\sigma$. Hence,

$$g = \varphi(g^{-1}g^\sigma) = \varphi(h^{-1}h^\sigma) = h.$$

Denote the group G regarded as a punctured symmetric space with the product $x.y = xy^{-1}x$ and the basic point 1 by the symbol G^+ . Obviously, the restriction of the homomorphism φ of N to G_σ induces a homomorphism $G_\sigma \rightarrow G^+$ of the punctured symmetric spaces. Let us an abstract symmetric space S isomorphic to $\varphi(G_\sigma)$, and let $P : S \rightarrow \varphi(G_\sigma)$ be the corresponding isomorphism. One can readily see that, for every coset Ig , where $g \in N^\varphi$, there is exactly one element $x \in S$ such that

$$Ig \cap \varphi(G_\sigma) = \{P_x\}.$$

(Here and everywhere below, for the sake of brevity, we use the symbol P_x instead of $P(x)$.) Hence, we can define a permutation representation of the group N^φ on S by setting $xg = z$ if $IP_xg = IP_z$ for $g \in N^\varphi$.

Theorem 1. *If a group G has the (σ, φ) -property, then the binary composition $xy = xP_y$ equips S with the structure of a Bol loop (of a Moufang loop provided that φ is an monomorphism).*

Proof. Let us fix a basic point e in the space S . Since every homomorphism of punctured spaces takes the basic point to the basic point, it follows that $P_e = 1$ and $P_{x^{-1}} = P_x^{-1}$. Therefore,

$$ex = x = xe \quad \text{and} \quad (yx)x^{-1} = y.$$

Further, $P_{x.y^{-1}} = P_xP_yP_x$. Therefore,

$$x.y^{-1} = (xy)x,$$

and hence

$$z((xy)x) = ((zx)y)x.$$

We have thus proved that S is a right Bol loop with the identity element. Let us now show that the equation $ax = b$ has a unique solution in the loop S . Indeed, if a solution exists, then it is unique, because

$$a^{-1}((ax)a) = xa \quad \text{and} \quad x = (a^{-1}(ba))a^{-1}.$$

On the other hand,

$$a((a^{-1}(ba))a^{-1}) = (ba)a^{-1} = b.$$

Thus, a solution of the equation $ax = b$ exists. Hence S is a Bol loop.

Let $\varphi : N \rightarrow G$ is an isomorphism of the group N into the group G . Then the mapping $L : S \rightarrow G^+$ defined by the relation $L_x = P_x^{-1}P_x^\sigma$ is an isomorphism of the punctured symmetric spaces. Therefore, $L_{x^{-1}} = L_x^{-1}$ and

$$(P_x L_y)^{-1}(P_x L_y)^\sigma = P_{(y^{-1}x)y^{-1}}^{-1}P_{(y^{-1}x)y^{-1}}^\sigma.$$

Hence,

$$IP_x L_y = IP_{(y^{-1}x)y^{-1}}.$$

Thus, the representation $x \rightarrow xL_y = (y^{-1}x)y^{-1}$ of the group N on S is defined. On the other hand,

$$P_{x^{-1}}^\sigma P_x = P_x P_{x^{-1}}^\sigma \quad \text{and} \quad yP_x L_x = yL_x P_x.$$

Therefore,

$$(x^{-1}(yx))x^{-1} = x^{-1}y \quad \text{and} \quad x(x^{-1}y) = y.$$

This implies that the Bol loop S is a loop with inversion, and thus a Moufang loop. This completes the proof of the theorem.

It is easy to prove that the generators P_x and $R_x = P_{x^{-1}}^\sigma$ of any (σ, φ) -group G satisfy the following identities:

$$P_e = 1, \quad R_e = 1, \tag{1}$$

$$P_{x^{-1}} = P_x^{-1}, \quad R_{x^{-1}} = R_x^{-1}, \tag{2}$$

$$P_{(xy)x} = P_x P_y P_x, \quad R_{(xy)x} = R_x R_y R_x, \tag{3}$$

$$P_x P_y P_{xy}^{-1} = R_x^{-1} R_y^{-1} R_{xy}. \tag{4}$$

Indeed, relations (1)–(3) immediately follow from Theorem 1. To prove relations (4), it suffice to note that

$$(P_x P_y)^{-1}(P_x P_y)^\sigma = P_{xy}^{-1}P_{xy}^\sigma.$$

Corollary 1. *If G is a group with the (σ, φ) -property, then the subgroup I of N^φ is generated by the elements $P_{x,y} = P_x P_y P_{xy}^{-1}$, there $x, y \in S$. In particular, $I = \{g \in N^\varphi \mid eg = e\}$.*

Proof. Let $H = \langle P_{x,y} \mid x, y \in S \rangle$ and $K = \{g \in N^\varphi \mid eg = e\}$, where e is the identity element of S . Obviously, $H \subseteq I \subseteq K$. Let us show that $K \subseteq H$. Indeed, $N^\varphi = \langle P_x \mid x \in S \rangle$. Therefore, every element of K can be represented as a word of the form $W = P_{x_1}P_{x_2} \dots P_{x_n}$. If $n = 1$, then $W = P_e \in H$. If $n > 1$, then $W = P_{x_1, x_2}P_{x_1 x_2} \dots P_{x_n}$, and the assertion is proved by an obvious induction on the length of the word W .

The (σ, φ) -group G and the Bol loop $S = S(G)$ constructed in Theorem 1 are said to be *associated*. If H is a subgroup of the group G , then we set

$$H_G = \bigcap_{g \in G} gHg^{-1}.$$

Obviously, H_G is the largest normal subgroup of G belonging to H . A subgroup H is said to be φ -admissible if

$$(N \cap H)^\varphi \subseteq N^\varphi \cap H.$$

A normal subgroup K of the group H is said to be σ -admissible if $K^\sigma = K$ and the induced automorphism $\bar{\sigma}$ of H/K is not identity. The σ -admissible and φ -admissible normal subgroup H of G is said to be (σ, φ) -admissible. If a group G contains no proper (σ, φ) -admissible normal subgroups, then this group is said to be (σ, φ) -simple.

Corollary 2. *Let H be a σ -admissible normal subgroup of a (σ, φ) -group G . The quotient group $\bar{G} = G/H$ has the $(\bar{\sigma}, \bar{\varphi})$ -property if and only if H is a φ -admissible subgroup of G . The loop \bar{S} associated with the group \bar{G} is a homomorphic image of $S = S(G)$, and $\bar{S} \simeq S$ if and only if $N^\varphi \cap H \subseteq I_{N^\varphi}$.*

Proof. Let $K = N \cap H$ and $M = N^\varphi \cap H$. Suppose $K^\varphi \subseteq M$. Since $N/K \simeq N^\varphi/K^\varphi$, it follows that the homomorphism $G \rightarrow \bar{G}$ induces the natural homomorphism $\bar{\varphi} : N/K \rightarrow N^\varphi/M$ with the kernel M/K^φ . It is obvious that \bar{G} is a $(\bar{\sigma}, \bar{\varphi})$ -group. Conversely, it follows from the existence of a homomorphism $\bar{\varphi}$ that its kernel satisfies the relation $\text{Ker}(\bar{\varphi}) = M/K^\varphi$. Thus, $K^\varphi \subseteq M$. To construct a homomorphism $S \rightarrow \bar{S}$, one can use the map $P_x \rightarrow P_x M$. In this case it is obvious that $S \simeq \bar{S}$ if and only if the elements of M trivially act on S , or equivalently, on the cosets of I . This completes the proof of the assertion.

Let S be a Bol loop associated with the (σ, φ) -group G , and let

$$T_x = L_x R_x^{-1}, \quad R_{x,y} = R_x R_y R_{xy}^{-1}, \quad L_{x,y} = L_x L_y L_{yx}^{-1},$$

where $L_x = P_x^{-1} R_x^{-1}$ and $x, y \in S$. Using the obvious identity $L_x^\sigma = L_x^{-1}$ it is easy to show that the set of all L_x generates a σ -invariant subgroup N in G . Consider the homomorphic images N^φ and \bar{I} of N and $N_L = \langle L_{x,y} \mid x, y \in S \rangle$ respectively arising as the map $\bar{\varphi} = \sigma\varphi\sigma$. Since $L_x^\sigma = L_x^{-1}$, $\varphi(L_x) = P_x$ and $P_x^\sigma = R_x^{-1}$, it follows that

$$N^\varphi = \langle R_x \mid x \in S \rangle, \quad \bar{I} = \langle R_{x,y} \mid x, y \in S \rangle,$$

and these groups are isomorphic to the groups N^φ and I respectively. Suppose \bar{S} is a symmetric space isomorphic to S . As above, we can define a binary composition on \bar{S} setting $xy = z$ if $\bar{I}R_xR_y = \bar{I}R_z$. It is readily seen that this composition equip \bar{S} with the structure of a Bol loop. Obviously, the loops \bar{S} and S are isomorphic. Finally, let

$$J = \langle T_x, R_{x,y}, L_{x,y} \mid x, y \in S \rangle.$$

Proposition 1. *If S is a Bol loop associated with the (σ, φ) -group G , then the multiplication group $Gr(S)$ of S is a homomorphic image of G . The kernel of the homomorphism is J_G .*

Proof. It can easily be checked that

$$JL_xR_y = JR_xR_y = JR_{xy} = JL_yL_x = JR_yL_x.$$

In addition, $L_x^{-1} = R_xP_x$ and $P_x = L_{x^{-1}}R_{x^{-1}}$. Therefore,

$$JL_x^{-1}R_y = JR_{x^{-1}y} \quad \text{and} \quad JR_xL_y^{-1} = JR_{(y^{-1}(xy))y^{-1}}.$$

Since any element of G is represented as a word of R_x and L_x , we prove, by induction, that every coset of J has the element R_x for some $x \in S$. Further, if $JR_x = JR_y$, then $R_{xy^{-1}} \in J$. On the other hand, the intersection $J \cap \bar{N}^\varphi = \bar{I}$. Therefore $R_{xy^{-1}} \in \bar{I}$, and hence $P_{xy^{-1}} \in I$. It is possible only if $x = y$. Hence, every coset of J in G has exactly one element R_x for every $x \in S$.

Thus, we can define a binary composition on the set of cosets of J setting $xy = z$ if $JR_xR_y = JR_z$ for all $x, y \in S$. Noting that $JR_x = JL_x$, we have

$$JR_yL_x = JR_{xy} \quad \text{or} \quad yL_x = xy.$$

Therefore the multiplication group $Gr(S)$ of S is a homomorphic image of G . The kernel of the homomorphism is the set of elements which induce the trivial permutation on S , or equivalently, on cosets of J . Obviously, this kernel is the subgroup J_G . This completes the proof of Proposition 1.

Suppose $J^\sigma \Sigma$ is a semidirect product of J^σ and $\Sigma = \langle \sigma \rangle$. Since the intersection $J^\sigma \cap N^\varphi = I$, it follows that $(J^\sigma \Sigma)_G \cap N^\varphi = I_{N^\varphi}$. Therefore, $(J^\sigma \Sigma)_G$ is the (σ, φ) -admissible normal subgroup of G . Using Corollary 2 to Theorem 1, we have

Corollary. *Let H be a normal (σ, φ) -admissible subgroup of the (σ, φ) -group G . Then the loops $S(G/H)$ and $S(G)$ are isomorphic if and only if $H \subseteq (J^\sigma \Sigma)_G$.*

3. Universality of the construction

Let S be a Bol loop. Denote by $\tilde{G}(S)$ the group presented by the generators P_x and R_x , where $x \in S$, and by the defining relations (1)–(4). Obviously, $\tilde{G}(S)$ admits an involutory automorphism σ defined by the mapping $P_x^\sigma = R_x^{-1}$ and $R_x^\sigma = P_x^{-1}$. One can present the group $\tilde{G}(S)$ in the form of a free product

$$\tilde{G}(S) = (A * B; A_1 = B_1, \psi)$$

of the groups

$$A = \langle P_x \mid x \in S \rangle \quad \text{and} \quad B = \langle R_x \mid x \in S \rangle$$

with the subgroups

$$A_1 = \langle P_x P_y P_{xy}^{-1} \mid x, y \in S \rangle \quad \text{and} \quad B_1 = \langle R_x^{-1} R_y^{-1} R_{xy} \mid x, y \in S \rangle$$

amalgamated according to the isomorphism $\psi : A \rightarrow B$. In this case, the automorphism σ of the group $\tilde{G}(S)$ can be defined by setting $\sigma = \psi$ on A and $\sigma = \psi^{-1}$ on B .

Theorem 2. *The group $\tilde{G}(S)$ is a (σ, φ) -group, and $S(\tilde{G}(S)) = S$. Any other (σ, φ) -group G such that $S(G) = S$ is a homomorphic image of the group $\tilde{G}(S)$.*

Proof. Let \tilde{N} be the subgroup of $\tilde{G} = \tilde{G}(S)$ generated by all elements of the form $L_x = P_x^{-1} R_x^{-1}$, where $x \in S$. Represent the relation (4) in the form

$$\begin{aligned} R_y L_x R_y^{-1} &= L_y^{-1} L_{xy}, \\ P_y^{-1} L_x P_y &= L_{xy} L_y^{-1}, \end{aligned}$$

and note that $L_x^\sigma = L_x^{-1}$. It is clear that \tilde{N} is a σ -invariant normal subgroup of the group \tilde{G} . Let W be an element of \tilde{G} represented as a word in the generators P_x and R_x , where $x \in S$. Then

$$\begin{aligned} (W P_x)^{-1} (W P_x)^\sigma &= P_x^{-1} W^{-1} W^\sigma P_x L_x, \\ (W R_x)^{-1} (W R_x)^\sigma &= R_x^{-1} W^{-1} W^\sigma R_x L_{x^{-1}}^{-1}. \end{aligned}$$

Using induction on the length of the word W , we can prove that $\tilde{N} = \langle g^{-1} g^\sigma \mid g \in \tilde{G} \rangle$. Since $A_1 P_x P_y = A_1 P_{xy}$, it is obvious that every right coset of the subgroup A_1 in the group A contains an element P_x for some $x \in S$.

Now, let us consider the homomorphism π of \tilde{G} onto A such that $\pi(a) = a$ and $\pi(b) = b^\sigma$ for all $a \in A$ and $b \in B$. It is obvious that $\text{Ker}(\pi) = \tilde{N}$. Therefore the crossings $\tilde{N} \cap A$ and $\tilde{N} \cap B$ are trivial. Hence (see [8]), \tilde{N} is a free group. On the other hand, there is a homomorphism of the group A onto the group of right multiplications of the loop S such that every generator P_x is mapped to the right multiplication operator on x in S . Therefore, $P_x \in A_1$ only if $x = e$, where e is the identity element of S , and $P_x P_y \in A_1$ only if $y = x^{-1}$. Hence the subgroup \tilde{N} is freely generated by the elements L_x , where $x \in S$ and $x \neq e$, and therefore the mapping $L_x \rightarrow P_x$ defines a homomorphism φ of \tilde{N} onto A . Thus, \tilde{G} is a (σ, φ) -group. It then follows from Theorem 1 that $S(\tilde{G}(S)) = S$.

Further, let G be another (σ, φ) -group such that $S(G) = S$. Then G has the set of generators P_x and R_x ($x \in S$) satisfying the relations (1)–(4), and therefore G is a homomorphic image of \tilde{G} . This completes the proof of the theorem.

Corollary 1. *For any Bol loop S , there is an associated group $G_0 = G_0(S)$ (which is unique up to isomorphism) such that any other group G associated with S has a homomorphic image equal to G_0 . Moreover, $(J_0^\sigma \Sigma)_{G_0} = 1$.*

Proof. Let $\tilde{G} = \tilde{G}(S)$ and $G_0 = \tilde{G}/(\tilde{J}^\sigma \Sigma)_{\tilde{G}}$. By Theorem 2, the groups G and G_0 are homomorphic images of \tilde{G} . Suppose $G = \tilde{G}/K$ for a (σ, φ) -admissible normal subgroup K . By Corollary to Proposition 1, K is contained in $(\tilde{J}^\sigma \Sigma)_{\tilde{G}}$. Since $(\tilde{J}^\sigma \Sigma)_{\tilde{G}}$ is a maximum (σ, φ) -admissible normal subgroup of \tilde{G} , it follows that G_0 is a minimal group associated with the loop S . Since $(\tilde{J}^\sigma \Sigma)_{\tilde{G}}$ is a prototype of $(J_0^\sigma \Sigma)_{G_0}$ in \tilde{G} , it follows that $(J_0^\sigma \Sigma)_{G_0} = 1$. The corollary is proved.

Corollary 2. *A Bol loop S is simple if and only if the associated group $G_0(S)$ is a (σ, φ) -simple. In addition, if the loop S is strongly simple, then the subgroup N_0^φ of the group $G_0(S)$ is a simple group.*

Proof. By Corollary 1 to Theorem 2, it follows that $(J_0^\sigma \Sigma)_{G_0} = 1$. Therefore, by Corollary to Proposition 1, if H is a proper (σ, φ) -admissible normal subgroup of $G_0(S)$, then $G_0(S)/H$ is a group with the (σ, φ) -property and its associated loop is a proper homomorphic image of S . Conversely, let S_1 be a proper normal subloop of S and $\phi : S \rightarrow S/S_1$ the induced homomorphism. Then the maps $P_x \rightarrow P_{\phi(x)}$ and $R_x \rightarrow R_{\phi(x)}$ can be extended to a homomorphism, provided that the image of an identity (1)–(4) in $\tilde{G}(S)$ is an identity in $\tilde{G}(S/S_1)$, and this readily verified. Therefore there exists the homomorphism $\tilde{G}(S) \rightarrow \tilde{G}(S/S_1)$. This homomorphism induces the homomorphism $G_0(S) \rightarrow G_0(S/S_1)$ with a (σ, φ) -admissible kernel. Therefore the group G_0 is not (σ, φ) -simple.

Further, suppose S is a strongly simple Bol loop and K is a normal subgroup of the group N_0^φ . Then the Proposition 1 required the condition $K \subseteq (J_0)_G \cap I_0$. On the other hand, the group I_0 is a set of fixed points of σ . Hence, $K^\sigma \subseteq (J_0^\sigma \Sigma)_{G_0}$. By Corollary 1 to Theorem 2, this implies that $K = 1$. The corollary is proved.

Denote by $Gr_r(S)$ and $Gr_l(S)$ the right and left multiplication groups of the Bol loop S respectively. The following assertion describes a structure of the multiplication group $Gr(S)$ in case if the loop S is strongly simple.

Proposition 2. *Let S be a strongly simple Bol loop, and H is a normal subgroup of $Gr(S)$. Then $Gr(S)$ is a φ -admissible group, and one of the following assertions are holds:*

- i) $Gr(S)$ is a simple group and S is a Moufang loop;
- ii) $H = Gr_l(S)$, and $Gr(S) = H \rtimes Gr_r(S)$ is a semidirect product of the subgroups;
- iii) $H \simeq Gr_r(S)$, and $Gr(S) = Gr_l(S) \times H$ is a direct product of the subgroups.

Proof. If $G_0 = G_0(S)$ is a simple group, then $N_0 = G_0$, and hence the subgroup $Ker(\varphi)$ of N_0 is a normal subgroup of the group G_0 . Obviously, $Ker(\varphi) \neq G_0$. Therefore, $Ker(\varphi) = 1$. Using Theorem 1, we prove that S is a Moufang loop.

Let H be a nonidentity proper normal subgroup of G_0 , and let $K = H \cap N_0$. If $K \not\subseteq Ker(\varphi)$, then K^φ is a nonidentity normal subgroup of N_0^φ . By Corollary 2 to Theorem 2, it is possible only if $K = N_0$. Therefore, $N_0 \subseteq H$. On the other hand,

$$G_0/H \simeq (G_0/N_0)/(H/N_0) \simeq N_0^\varphi/(H/N_0).$$

Since the group N_0^φ is simple, it follows that $H = N_0$. In addition, it is obvious that $N_0 \cap N_0^\varphi = 1$. On the other hand, it is clear that the group G_0 is generated by the subgroups N_0 and N_0^φ . Therefore G_0 is a semidirect product of the subgroups.

Let $K \subseteq \text{Ker}(\varphi)$. Since $\text{Ker}(\varphi)_{G_0}$ is the largest normal subgroup of G_0 belonging to $\text{Ker}(\varphi)$, it follows that $K \subseteq \text{Ker}(\varphi)_{G_0}$. On the other hand, HN_0 is a normal subgroup of G_0 . As was shown above, it is not a proper normal subgroup of G_0 . Hence, $HN_0 = G_0$.

Thus, the quotient group $G_0/\text{Ker}(\varphi)_{G_0}$ is a simple or semisimple group. On the other hand, Corollary 1 to Theorem 1 claims that $I_0 \subseteq J_0^\varphi$. Therefore $\text{Ker}(\varphi) \subseteq J_0$, and hence $\text{Ker}(\varphi)_{G_0} \subseteq (J_0)_{G_0}$. Using Proposition 1, we prove the assertion.

4. Finite strongly simple Bol loops

Recall (see [8]) that a group G is said to be *residually finite* if the intersection of all its normal subgroups of finite index is the identity group. A group G is residually finite if and only if, for every element $g \in G$, $g \neq 1$ there is a homomorphism of G into a finite group that takes g to a nonidentity element. As is known (see [9]), in particular, every free product of two finite groups with an amalgamated subgroup is a residually finite group.

Theorem 3. *If S is a finite Bol loop of order n , then $\tilde{G} = \tilde{G}(S)$ is a residually finite group. In particular, the group G_0 is finite.*

Proof. It is obvious that the group \tilde{N}^φ admits an involutory automorphism θ such that $P_x^\theta = P_x^{-1}$ for all $x \in S$. Consider the semidirect products $\tilde{N}^\varphi \langle \theta \rangle$. By the relations (1)–(3), it follows that

$$(P_x \theta)^\theta = P_{x^{-1}} \theta, \quad (P_x \theta)^{P_y} = P_{(y^{-1}x)y^{-1}} \theta.$$

These relations imply that the group $\tilde{N}^\varphi \langle \theta \rangle$ is generated by a finite set of elements each of which is of finite order and has finitely many conjugate elements.

Let $|S| = n$. Then $X = \{P_{x_1} \theta, \dots, P_{x_n} \theta\}$ is the set of generators of the group $\tilde{N}^\varphi \langle \theta \rangle$ and of the elements conjugate to the generators. Let us show that every element of $\tilde{N}^\varphi \langle \theta \rangle$ can be represented by a word W of length $\leq n$. Indeed, $(P_{x_i} \theta)^2 = 1$. Therefore,

$$W = (\dots P_{x_i} \theta W_1 P_{x_i} \theta \dots) = (\dots (P_{x_i} \theta)^{-1} W_1 P_{x_i} \theta \dots).$$

Since, when conjugating $P_{x_j} \theta$ by $P_{x_i} \theta$, we obtain one of the elements in the list X again, it follows that the length of the word W can be reduced. After finitely steps we obtain a representation of W in the form of a word of length $\leq n$. It is easily shown that the order

$$|\tilde{N}^\varphi| \leq \frac{1}{2} \sum_{k=1}^n \frac{n!}{(n-k)!} < \frac{1}{2} en!,$$

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where e is the base of the natural logarithms. Since every free product of two finite groups with an amalgamated subgroup is a residually finite group, it follows that the first part of the assertion is proved.

Thus, N_0 is a finite-index subgroup of G_0 , and $\text{Ker}(\varphi)$ is a finite-index subgroup of N_0 . This implies that $K = \text{Ker}(\varphi)_{G_0}$ is a finite-index normal subgroup of G_0 . Hence, G_0 is a finite group. This completes the proof of the theorem.

By analogy with the notion of an algebraic group we define an *algebraic Bol loop*. One is a Bol loop S equipped with an algebraic variety structure such that the product $S \times S \rightarrow S$ and the inversion $S \rightarrow S$ are regular mappings (morphisms) of the algebraic varieties.

Theorem 4. *If S is an connected algebraic strongly simple Bol loop, and its multiplication group $\text{Gr}(S)$ is an algebraic group, then S is a Moufang loop. In particular, every connected finite strongly simple Bol loop is Moufang.*

Proof. As is well known (see e.g. [10,11]), there exists two basic type of connected algebraic groups: *Abelian varieties* and *linear (or affine) algebraic groups*. Any Abelian variety is an Abelian group. Any linear algebraic group are a subgroup of the group $GL(n, k)$. These two classes of groups have the trivial intersection. It is known that:

- i) any connected algebraic group G has unique normal linear algebraic subgroup H such that the quotient group G/H is an Abelian variety;
- ii) any linear algebraic group G has a maximum connected solvable normal subgroup (a *radical*) K such that the quotient group G/K is semisimple.

Let S be an connected algebraic strongly simple Bol loop, and $\text{Gr}(S)$ be an algebraic group. Obviously, $\text{Gr}_r(S)$ and $\text{Gr}_r(S)^\sigma$ are connected subgroups of $\text{Gr}(S)$. Let $G(M)$ be an intersection of all closed subgroups of $\text{Gr}(S)$ containing the set $M = \text{Gr}_r(S) \cup \text{Gr}_r(S)^\sigma$. Then (see [10]) $G(M)$ is a connected algebraic subgroup of $\text{Gr}(S)$. Since the set M generates the group $\text{Gr}(S)$, it follows that $\text{Gr}(S)$ is a connected algebraic group.

Suppose H is a normal linear algebraic subgroup of $\text{Gr}(S)$ such that $\text{Gr}(S)/H$ is an Abelian manifold. It follows from Propositions 2 that either $H = 1$ or $H = \text{Gr}_l(S)$ or $H \simeq \text{Gr}_r(S)$ or $H = \text{Gr}(S)$. If $H = 1$, then $\text{Gr}(S)$ is an Abelian group, and hence S is a simple Abelian group. If $H = \text{Gr}_l(S)$ or $H \simeq \text{Gr}_r(S)$, then $\text{Gr}_r(S)$ is a linear algebraic group and an Abelian variety simultaneously. It is possible only if S is the identity group.

Let $\text{Gr}(S)$ be a linear algebraic group, and let K be a radical of $\text{Gr}(S)$. If $K \neq 1$, then either $K = \text{Gr}(S)$ or $K = \text{Gr}_l(S)$ or $K \simeq \text{Gr}_r(S)$. In any case $\text{Gr}_r(S)$ is a solvable group. Since $\text{Gr}_r(S)$ is a simple group, it follows that it is an Abelian group, and hence S is a simple commutative Moufang loop. If $K = 1$, then $\text{Gr}(S)$ is a simple group, and S is a Moufang loop.

Finally, let S be an finite connected algebraic strongly simple Bol loop. It follows from Theorem 3 that the group $\text{Gr}(S)$ is finite, and hence one may be embedded into the linear algebraic group $GL(n, k)$. As above, this implies that $\text{Gr}(S)$ is a connected finite group, and hence S is a finite simple Moufang loop. This completes the proof of Theorem 4.

Corollary. *If the right multiplication group $Gr_r(S)$ of a Bol loop S is a finite simple group of Lie type, then S is a finite simple Moufang loop.*

Proof. Obviously, S is a finite loop. Therefore $Gr(S)$ is a finite group, and hence one may be embedded into $GL(n, k)$. As in the previous theorem, this implies that $Gr(S)$ is a connected algebraic group, and hence S is a finite simple Moufang loop.

Remark 1. Apparently, Theorem 4 may be extended to the class of all connected finite simple Bol loop. The key to the proof is the assertion that the subgroup N_0° for such loop is a simple group. However, we have not any proof of the assertion.

Remark 2. It is known [12] that the group $Aut(S)$ of all invertible morphisms of a loop S into itself has a natural structure of an algebraic group. Obviously, $Gr(S)$ is a subgroup of $Aut(S)$. However, the group $Aut(S)$ can have infinite number of connected components, and hence the assertions i) and ii) for $Aut(S)$ may not be true. The example that was constructed in [6] shows that there exists a strongly simple non-Moufang Bol loop with a non-algebraic multiplication group.

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